带有非线性边界和扰动输入的波动方程的输出反馈控制

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摘要: 无穷维系统的输出反馈控制是控制理论中重要的研究课题, 相对于线性边界输入而言, 非线性边界条件更多应用于实际的数学模型中, 容易引起各种不同的动力学行为, 如混沌声振动、倍周期分岔、方波等. 本文研究了左端具有非线性位移边界条件, 右端带有总扰动输入的一维波动方程的输出反馈镇定问题. 首先, 利用算子半群理论证明了开环系统的适定性; 其次, 由于内部非线性项和外部扰动的存在, 通过构造无穷维扰动估计器, 证明了该估计器能够实时在线估计总扰动; 紧接着, 借助于原系统的量测输出信号设计状态观测器, 构造输出反馈控制器并得到了闭环系统; 最后, 证明了该闭环系统的适定性和渐近稳定性.

关键词: 波动方程; 非线性边界条件; 干扰估计器; 输出反馈稳定

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Output feedback control for wave equations with nonlinear boundary condition and disturbance inputs

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Abstract: The output feedback control of infinite dimensional systems is an important research topic in control theory. Compared with linear boundary input, nonlinear boundary conditions are more applied to practical mathematical models, which can cause various dynamic behaviors, such as chaotic acoustic vibration, period-doubling bifurcation, square wave, and so on. In this paper, the output feedback stability problem of one dimensional wave equation with nonlinear displacement boundary condition at left end and total disturbance input at right end is studied. Firstly, the well-posedness of open loop systems is proved by using operator semigroup theory. Secondly, due to the existence of internal nonlinear terms and external disturbances, we prove that the estimator can estimate total disturbances by constructing an infinite-dimensional disturbance estimator. Then, the state observer is designed by means of the measured output signal of the original system, and the stability controller is constructed and the closed-loop system is obtained. Finally, the well-posedness and asymptotic stability of the closed-loop system are proved.

Key words: wave equation; nonlinear boundary condition; disturbance estimator; output feedback stabilization

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1 引言

分布参数系统的控制问题一直是控制理论界所关注和研究的核心论题之一, 由偏微分方程所描述的控制系统是分布参数系统的一个重要部分. 本文主要研究由双曲型波动方程所描述的控制系统的输出反馈问题. 在实际工程和应用科学中, 大多数现象都是用线性或非线性波动方程来模拟的, 比如流体力学、光学、电磁学、量子力学等[1–2]. 因此, 近年来对一维波动方程输出反馈稳定性的研究越来越多[3–4]. 然而, 大多数输出反馈控制都是利用速度边界测量来设计的. 另一方面, 由于速度传感器通常重量大, 成本高, 直接测量速度比较困难. 与量测速度相比, 边界位移的测量是相对容易实现的[5]. 另外, 相对于线性边界输入而言, 非线性边界条件更多应用于实际的数学模型中,

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可以引起各种不同的动力学行为, 如混沌声振动、倍周期分岔、方波等[6–7].

从实际应用角度来看, 很多物理现象、自然规律本质上都是通过偏微分方程来描述的. 由于时空因素和外部环境的影响, 对于边界具有非线性项和扰动的波动方程的研究变得更加复杂且具有挑战性, 很多文献对这类问题进行了研究[8–9]. 在文献[10]中, 作者应用backstepping方法研究了具有Direchlet边界条件的一维波动方程的输出反馈指数稳定性, 提出了一种新的输出反馈控制律, 但是该问题没有讨论边界带有干扰项. 在文献[11]中, 作者采用积分滑模控制的方法研究了边界带有扰动的非线性边界输入的一维波动方程的稳定性, 并且证明了闭环系统的适定性, 但是该方法设计的控制器具有非常高的比例增益, 并且在实际工程中往往会引起颤抖现象. 在文献[12]中, 作者采用了3个量测信号研究了左边边界带有速度立方项$w_t^3(0,t)$输入的波动方程的输出反馈问题, 错误的使用了$w_t(0,t)$在状态空间$L^2(0,1)$中的有界性, 得到了系统的稳定性. 基于上述研究, 本论文仅基于边界位移信号, 研究了一端具有位移非线性项($w^3(0,t)$), 另一端边界具有控制匹配扰动输入的一维波动方程的输出反馈稳定性问题.

本文研究如下具有非线性位移输入和带有Neumann边界控制匹配扰动的一维弦振动方程的稳定性, 即

$$\begin{cases} w_{tt}(x,t)=w_{xx}(x,t), & 0<x<1,\\ \quad t>0, & \\ w_x(0,t)=-qw_t(0,t)+pw^3(0,t), & t\geqslant 0,\\ w_x(1,t)=u(t)+d(t)+f(w(\cdot,t),w_t(\cdot,t)), & \\ & t\geqslant 0,\\ w(x,0)=w_0(x), w_t(x,0)=w_1(x), & 0\leqslant x\leqslant 1,\\ y_{\mathrm{m}}(t)=(w(0,t),w(1,t)), & t\geqslant 0, \end{cases} \tag{1}$$

其中: 常数$p>0$, $0<q<1$; $w(\cdot,t)$是系统的状态; $u(t)$代表控制输入; $y_{\mathrm{m}}(t)$表示量测输出; $f: H^1(0,1)\times L^2(0,1)\to\mathbb{R}$是未知的内部非线性不确定项; $d(t)$是未知的外部扰动, 假设$d\in L^\infty(0,\infty)$或者$d\in L^2(0,\infty)$. 令$F(t)=d(t)+f(w(\cdot,t),w_t(\cdot,t))$称为总扰动. 当$p=q=0$时, 系统(1)是一个保守系统, 表示一端受控制, 另一端自由的振动弦. 在文献[4]中, 作者研究了非同位观测一维保守系统的稳定性. 当$p=0$, $q>0$时, 无控制系统(1)变为一个经典的反稳定系统, 文献[13]中通过backstepping方法进行了稳定性分析.

本文在状态空间$\mathcal{H}=H^1(0,1)\times L^2(0,1)$中研究系统(1), 定义对应的内积如下:

$$\langle (f_1,g_1),(f_2,g_2)\rangle_{\mathcal{H}}=\int_0^1 (f_1'(x)\bar{f}_2'(x)+g_1(x)\bar{g}_2(x))\mathrm{d}x+f_1(0)\bar{f}_2(0),\quad \forall (f_i,g_i)\in\mathcal{H},\ i=1,2.$$

本文结构安排如下: 第2部分研究了开环系统的适定性问题; 第3部分构造无穷维扰动估计器估计扰动; 第4部分基于干扰估计器设计了状态观测器; 第5部分通过设计输出反馈稳定控制器, 并对闭环系统的适定性与稳定性进行了分析; 第6部分为结论部分.

2 开环系统的适定性

定义算子$\mathcal{A}$: $D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$和$\mathcal{B}_1,\mathcal{B}_2$: $\mathbb{R}\to\mathcal{H}$, 如下:

$$\begin{cases} \mathcal{A}(f,g)=(g,f''),\\ D(\mathcal{A})=\{(f,g)\in H^2(0,1)\times H^1(0,1)|\\ \qquad f'(0)=f(0),\ f'(1)=0\},\\ \mathcal{B}_1=(0,-\delta(x)),\ \mathcal{B}_2=(0,\delta(x-1)), \end{cases} \tag{2}$$

其中$\delta(\cdot)$是Dirac分布, 通过简单计算可知$\mathcal{A}^*=-\mathcal{A}$. 系统(1)便可以描述成如下抽象发展方程形式:

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(\cdot,t),w_t(\cdot,t))=\mathcal{A}(w(\cdot,t),w_t(\cdot,t))+\mathcal{B}_1H(\cdot,t)+\mathcal{B}_2(f(w(\cdot,t),w_t(\cdot,t))+u(t)+d(t)), \tag{3}$$

其中: H: $\mathcal{H}\to\mathbb{R}$, $H(w(\cdot,t),w_t(\cdot,t)):=pw(0,t)^3-qw_t(0,t)-w(0,t)$. 在代数意义上, 系统(1)和系统(3)是等价的[14]. 下面的命题1通过压缩映像原理证明了开环系统(1)解的存在性和唯一性.

命题 1 由(2)所定义的算子$\mathcal{A}$在空间$\mathcal{H}$上生成一个C_0半群, $\mathcal{B}_1,\mathcal{B}_2$是$\mathrm{e}^{\mathcal{A}t}$的可允许性算子. 假设$f$: $\mathcal{H}\to\mathbb{R}$是连续的, 并且满足局部Lipschitz条件, $f(0,0)=0$, $w_t(0,0)=0$, 那么对于任何$(w_0,w_1)\in\mathcal{H}$, $u\in L^2_{\mathrm{loc}}[0,\infty)$, 对于某些$T>0$, 当$t\in[0,T]$, 系统(1)存在唯一的局部解, 满足$(w(.,t),w_t(.,t))\in C(0,\infty;\mathcal{H})$, 即

$$(w(\cdot,t),w_t(\cdot,t))=\mathrm{e}^{\mathcal{A}t}(w_0(\cdot),w_1(\cdot))+\int_0^t \mathrm{e}^{\mathcal{A}(t-s)}\mathcal{B}_1H(\cdot,s)\mathrm{d}s+\int_0^t \mathrm{e}^{\mathcal{A}(t-s)}\mathcal{B}_2((f(w\cdot,s),w_t(\cdot,s))+u(s)+d(s))\mathrm{d}s, \tag{4}$$

进一步, 如果$f:\mathcal{H}\to\mathbb{R}$和$H:\mathcal{H}\to\mathbb{R}$满足一致Lipschitz条件或者它们有界, 那么系统(1)在$T=\infty$时存在唯一全局解$(w,w_t)\in C(0,\infty;\mathcal{H})$, 且满足式(4).

证 由文献[12]可知, $\mathcal{A}$生成了一个酉群, $\mathcal{B}_1,\mathcal{B}_2$是$\mathrm{e}^{\mathcal{A}t}$的可允许性算子. 因此, 对于给定的$T>0$, u, $d\in L^2_{\mathrm{loc}}[0,\infty)$, 可以得到

$$\int_0^t \mathrm{e}^{\mathcal{A}(t-s)}\mathcal{B}_2(u(s)+d(s))\mathrm{d}s\in C(0,T;\mathcal{H}), \tag{5}$$

接下来, 通过压缩映像原理证明解的唯一性. 对于任

意初值$[w_0\ \ w_1]^{\mathrm T}\in\mathcal H$, 令$(\delta_1(t),\delta_2(t))=\mathrm e^{\mathcal At}(w_0(\cdot),w_1(\cdot))$. $\forall\sigma>0$, 定义一个集合$\varDelta_t$, 即

$$\varDelta_t=\{(\nu,\rho)\in\mathcal H,\|(\nu,\rho)-(\delta_1(t),\delta_2(t))\|_{\mathcal H}\leqslant\sigma\},$$

因为f和$H(w(\cdot,t),w_t(\cdot,t))$满足局部Lipschitz条件, 那么存在常数$L_1^\sigma,L_2^\sigma$(不依赖于$t$)满足

$$\begin{aligned}&|f(\phi_1(\cdot,t),\psi_1(\cdot,t))-f(\phi_2(\cdot,t),\psi_2(\cdot,t))|\leqslant\\&L_1^\sigma\|(\phi_1(\cdot,t),\psi_1(\cdot,t))-(\phi_2(\cdot,t),\psi_2(\cdot,t))\|_{\mathcal H},\\&|H(\phi_1(\cdot,t),\psi_1(\cdot,t))-H(\phi_2(\cdot,t),\psi_2(\cdot,t))|\leqslant\\&L_2^\sigma\|(\phi_1(\cdot,t),\psi_1(\cdot,t))-(\phi_2(\cdot,t),\psi_2(\cdot,t))\|_{\mathcal H},\end{aligned}$$

由$\mathcal B_1,\mathcal B_2$的可允许性知, 对于所有的$t>0$, 存在$C_{1t}$, $C_{2t}>0$满足

$$\begin{aligned}&\|\int_0^t\mathrm e^{\mathcal A(t-s)}\mathcal B_1(H(\phi_1(\cdot,s),\psi_1(\cdot,s))-\\&H(\phi_2(\cdot,s),\psi_2(\cdot,s)))\mathrm ds\|_{\mathcal H}\leqslant\\&C_{1t}t\|H(\phi_1(\cdot,s),\psi_1(\cdot,s))-\\&H(\phi_2(\cdot,s),\psi_2(\cdot,s))\|_{L^\infty(0,t)},\end{aligned}\tag{6}$$

$$\begin{aligned}&\|\int_0^t\mathrm e^{\mathcal A(t-s)}\mathcal B_2(f(\phi_1(\cdot,s),\psi_1(\cdot,s))-\\&f(\phi_2(\cdot,s),\psi_2(\cdot,s)))\mathrm ds\|_{\mathcal H}\leqslant\\&C_{2t}t\|f(\phi_1(\cdot,s),\psi_1(\cdot,s))-\\&f(\phi_2(\cdot,s),\psi_2(\cdot,s))\|_{L^\infty(0,t)},\end{aligned}\tag{7}$$

由文献[15]的命题2.3可得, C_{1t},C_{2t}是关于t不减的. 设$t\leqslant T$, 那么$C_{1t}\leqslant C_{1T}$, $C_{2t}\leqslant C_{2T}$. 令$L^\sigma=\max\{L_1^\sigma,L_2^\sigma\}$, $C_t=\max\{C_{1t},C_{2t}\}$. 设$T>0$, 满足$2C_tTL^\sigma<1$且

$$\begin{aligned}&\sigma+\|\mathrm e^{\mathcal At}(w_0(\cdot),w_1(\cdot))+\\&\int_0^t\mathrm e^{\mathcal A(t-s)}\mathcal B_2(u(s)+d(s))\mathrm ds\|_{C(0,T;\mathcal H)}\leqslant\\&\frac{\sigma}{2C_tTL^\sigma}.\end{aligned}\tag{8}$$

令$\varTheta=\{(\phi(\cdot,t),\psi(\cdot,t))\in C(0,T;\mathcal H)$: $\phi(\cdot,0)=w_0(\cdot),\psi(\cdot,0)=w_1(\cdot),\|(\phi(\cdot,t),\psi(\cdot,t))-\mathrm e^{\mathcal At}(w_0(\cdot),w_1(\cdot))-\int_0^t\mathrm e^{\mathcal A(t-s)}\mathcal B_2(u(s)+d(s))\mathrm ds\|_{\mathcal H}\leqslant\sigma\}$是空间$C(0,T;\mathcal H)$的一个闭子空间. 定义非线性映射$F$: $\varTheta\to C(0,T;\mathcal H)$, 如下:

$$\begin{aligned}&F(\phi(\cdot,t),\psi(\cdot,t))=\\&\mathrm e^{\mathcal At}(w_0(\cdot),w_1(\cdot))+\int_0^t\mathrm e^{\mathcal A(t-s)}\mathcal B_2(u(s)+d(s))\mathrm ds+\\&\int_0^t\mathrm e^{\mathcal A(t-s)}\mathcal B_1H(\phi(\cdot,s),\psi(\cdot,s))\mathrm ds+\\&\int_0^t\mathrm e^{\mathcal A(t-s)}\mathcal B_2f(\phi(\cdot,s),\psi(\cdot,s))\mathrm ds.\end{aligned}\tag{9}$$

容易证明F是$C(0,T;\mathcal H)$上的一个压缩映射, $\forall(\phi_1(\cdot,s),\psi_1(\cdot,s)),(\phi_2(\cdot,s),\psi_2(\cdot,s))\in C(0,T;\mathcal H)$, 有

$$\begin{aligned}&\|\int_0^t\mathrm e^{\mathcal A(t-s)}\mathcal B_1(H(\phi_1(\cdot,s),\psi_1(\cdot,s))-\\&H(\phi_2(\cdot,s),\psi_2(\cdot,s)))\mathrm ds\|_{\mathcal H}\leqslant\\&C_{1t}TL_2^\sigma\|(\phi_1(\cdot,s),\psi_1(\cdot,s))-\\&(\phi_2(\cdot,s),\psi_2(\cdot,s))\|_{C(0,T;\mathcal H)},\end{aligned}\tag{10}$$

同理, 可得

$$\begin{aligned}&\|\int_0^t\mathrm e^{\mathcal A(t-s)}\mathcal B_2(f(\phi_1(\cdot,s),\psi_1(\cdot,s))-\\&f(\phi_2(\cdot,s),\psi_2(\cdot,s)))\mathrm ds\|_{\mathcal H}\leqslant\\&C_{2t}TL_1^\sigma\|(\phi_1(\cdot,s),\psi_1(\cdot,s))-\\&(\phi_2(\cdot,s),\psi_2(\cdot,s))\|_{C(0,T;\mathcal H)}.\end{aligned}\tag{11}$$

由式(10)–(11), $\forall(\phi_1(\cdot,s),\psi_1(\cdot,s))$, $(\phi_2(\cdot,s),\psi_2(\cdot,s))\in C(0,T;\mathcal H)$可得

$$\begin{aligned}&\|F(\phi_1(\cdot,s),\psi_1(\cdot,s))-F(\phi_2(\cdot,s),\psi_2(\cdot,s))\|_{\mathcal H}\leqslant\\&2C_tTL^\sigma\|(\phi_1(\cdot,s),\psi_1(\cdot,s))-\\&(\phi_2(\cdot,s),\psi_2(\cdot,s))\|_{C(0,T;\mathcal H)}.\end{aligned}\tag{12}$$

因为$2C_tTL^\sigma<1$, 所以F是严格压缩的. 令$(\phi_2(\cdot,s),\psi_2\cdot,s))=(0,0)$, 可证$F\varTheta\subset\varTheta$. 由压缩映像原理可知, 式(9)有唯一的不动点$(w(\cdot,t),w_t(\cdot,t))\in C(0,\infty;\mathcal H)$, 且是系统(3)的解. 进一步, 由参考文献[16]的命题1.1可得全局解的存在性. 证毕.

为了方便讨论, 接下来的系统把x和t的取值范围都省略掉.

3 干扰估计器设计和分析

本节借助系统(1)的输出量测信号构造总扰动估计器如下:

$$\begin{cases}v_{xx}(x,t)=v_{tt}(x,t),\\v_x(0,t)=-qv_t(0,t)+pv^3(0,t)+\\\qquad c_1(v(0,t)-w(0,t)),\\v_x(1,t)=u(t)+W_x(1,t),\\v(x,0)=v_0(x),\ v_t(x,0)=v_1(x),\\z_{xx}(x,t)=z_{tt}(x,t),\\z_x(0,t)=\dfrac{c_0-q}{1-c_0}z_t(0,t)+\dfrac{c_1}{1-c_0}z(0,t)+h(\cdot,t),\\z(1,t)=w(1,t)-v(1,t)+W(1,t),\\z(x,0)=z_0(x),\ z_t(x,0)=z_1(x),\\W_t(x,t)+W_x(x,t)=0,\\W(0,t)=-c_0(w(0,t)-v(0,t)),\\W(x,0)=W_0(x),\end{cases}\tag{13}$$

其中: c_0,c_1是两个正的设计参数, $h(\cdot,t)=p(w^3(0,t)-v^3(0,t))$, $(v_0,v_1,z_0,z_1,W_0)\in\mathcal H^2\times H^1(0,1)$是扰动估计量的初始状态. 令

$$\widehat v(x,t)=w(x,t)-v(x,t),$$

则

$$\begin{cases}\widehat{v}_{xx}(x,t)=\widehat{v}_{tt}(x,t),\\ \widehat{v}_x(0,t)=-q\widehat{v}_t(0,t)+c_1\widehat{v}(0,t)+h(\cdot,t),\\ \widehat{v}_x(1,t)=F(t)-W_x(1,t),\\ W_t(x,t)+W_x(x,t)=0,\\ W(0,t)=-c_0(w(0,t)-v(0,t)).\end{cases}\tag{14}$$

令$\widetilde{v}(x,t)=\widehat{v}(x,t)+W(x,t)$, 容易验证$(\widetilde{v}(x,t), W(x,t))$满足

$$\begin{cases}\widetilde{v}_{xx}(x,t)=\widetilde{v}_{tt}(x,t),\\ \widetilde{v}_x(0,t)=\dfrac{c_0-q}{1-c_0}\widetilde{v}_t(0,t)+\dfrac{c_1}{1-c_0}\widetilde{v}(0,t)+h(\cdot,t),\\ \widetilde{v}_x(1,t)=F(t),\\ W_t(x,t)+W_x(x,t)=0,\\ W(0,t)=\dfrac{-c_0}{1-c_0}\widetilde{v}(0,t),\end{cases}\tag{15}$$

其中: $\widetilde{v}(x,0)=\widehat{v}(x,0)+W(x,0)$, $\widetilde{v}_t(x,0)=\widehat{v}_t(x,0)-W_x(x,0)$是初始时刻的值, 在Hilbert空间$\mathcal{H}=H^1(0,1)\times L^2(0,1)$中研究系统(15), 定义内积如下:

$$\langle(f_1,g_1),(f_2,g_2)\rangle=\int_0^1(f_1'(x)\bar{f}_2'(x)+g_1(x)\bar{g}_2(x))\mathrm{d}x+\frac{c_1}{1-c_0}f_1(0)\bar{f}_2(0),\ \forall(f_i,\ g_i)\in\mathcal{H},\ i=1,2.\tag{16}$$

引理 1 假设$\dfrac{c_0-q}{1-c_0}>0$, $\dfrac{c_1}{1-c_0}>0$, $d\in L^\infty(0,\infty)$或$d\in L^2(0,\infty)$, $f: H^1(0,1)\times L^2(0,1)\to\mathbb{R}$是连续的, 那么系统(1)存在唯一的解$(w,w_t)\in C(0,\infty;\mathcal{H})$, 并且$(w,w_t),(v,v_t)\in C(0,\infty;\mathcal{H})$是有界的. 对于任意初始状态$(\widehat{v}_0,\widehat{v}_1,W_0)\in H^1(0,1)\times L^2(0,1)$, 且满足相容性条件$W_0(0)=-c_0\widehat{v}_0(0)$, 系统(15)存在唯一的解$(\widehat{v},\widehat{v}_t,W)\in C(0,\infty;\mathcal{H}\times H^1(0,1))$, 并且有

$$\sup_{t\geqslant0}\|(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t),W(\cdot,t))\|_{\mathcal{H}\times H^1(0,1)}<\infty.\tag{17}$$

进一步, 若$f(w,w_t)\in L^2(0,\infty)$或者$\lim\limits_{t\to\infty}|f(w,w_t)|=0$, $d\in L^2(0,\infty)$, 有

$$\lim_{t\to\infty}\|(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t),W(\cdot,t))\|_{\mathcal{H}\times H^1(0,1)}=0.\tag{18}$$

证 首先, 考虑变换后的系统(15), 该系统的初始状态为$\widetilde{v}(x,0)=\widehat{v}(x,0)+W(x,0)$, $\widetilde{v}_t(x,0)=\widehat{v}_t(x,0)-W_x(x,0)$, 由引理假设可知$\widetilde{v}(x,0),\widetilde{v}_t(x,0)\in\mathcal{H}$. 定义算子$\mathcal{A}_{\widetilde{v}}$和$\mathcal{B}_1,\mathcal{B}_2$, 如下:

$$\begin{cases}\mathcal{A}_{\widetilde{v}}(f,g)=(g,f''),\\ D(\mathcal{A}_{\widetilde{v}})=\{(f,g)\in H^2(0,1)\times H^1(0,1)|\\ \qquad f'(0)=\dfrac{c_0-q}{1-c_0}g(0)+\dfrac{c_1}{1-c_0}f(0),\\ \qquad f'(1)=0\},\\ \mathcal{B}_1=(0,-\delta(x)),\ \mathcal{B}_2=(0,\delta(x-1)),\end{cases}\tag{19}$$

其中$\delta(\cdot)$是Dirac分布, 系统(15)可以写成如下形式:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{v}(\cdot,t),\widetilde{v}_t(\cdot,t))=\mathcal{A}_{\widetilde{v}}(\widetilde{v}(\cdot,t),\widetilde{v}_t(\cdot,t))+\mathcal{B}_1h(\cdot,t)+\mathcal{B}_2F(t),\tag{20}$$

显然, $\mathcal{A}_{\widetilde{v}}$生成指数稳定的算子半群$\mathrm{e}^{\mathcal{A}_{\widetilde{v}}t}$, 且$\mathcal{B}_1,\mathcal{B}_2$是$\mathrm{e}^{\mathcal{A}_{\widetilde{v}}t}$的可允许算子. 由于$f:\mathcal{H}\to R$是连续函数, $(w,w_t)\in C(0,\infty)$是有界的, 可得$f(w,w_t)\in L^\infty(0,\infty)$, $h(\cdot,t)=p(w^3(0,t)-v^3(0,t))\in L^2(0,\infty)$, $d\in L^\infty(0,\infty)$或$d\in L^2(0,\infty)$, 那么由文献[16]中引理2.1可知, 存在常数$M_1>0$, 系统(15)中的$\widetilde{v}$存在唯一的有界解

$$\sup_{t\geqslant0}\|(\widetilde{v}(\cdot,t),\widetilde{v}_t(\cdot,t))\|_{\mathcal{H}}<M_1.\tag{21}$$

接下来, 证明$\|W(\cdot,t)\|_{H^1(0,1)}$的有界性. 为此先证明对于所有的$t\geqslant1$, 有

$$\int_0^1\widetilde{v}_t(0,t-x)\mathrm{d}x\leqslant3\max_{\tau\in[t-1,t]}\|\widetilde{v}(x,\tau),\widetilde{v}_t(x,\tau)\|^2_{\mathcal{H}}.\tag{22}$$

定义辅助函数

$$\psi(t)=2\int_0^1(x-1)\widetilde{v}_t(x,t)\widetilde{v}_x(x,t)\mathrm{d}x,\tag{23}$$

容易证明

$$|\psi(t)|\leqslant\|\widetilde{v}(x,t),\widetilde{v}_t(x,t)\|^2_{\mathcal{H}}.\tag{24}$$

对$\psi(t)$求导得

$$\psi(t)=\widetilde{v}_t(0,t)^2+\widetilde{v}_x(0,t)^2-\int_0^1|\widetilde{v}_t(x,t)|^2+|\widetilde{v}_x(x,t)|^2\mathrm{d}x\geqslant\widetilde{v}_t(0,t)^2-\int_0^1|\widetilde{v}_t(x,t)|^2+|\widetilde{v}_x(x,t)|^2\mathrm{d}x.\tag{25}$$

因此, 对所有的$t\geqslant1$, 有

$$\int_{t-1}^t\widetilde{v}_t(0,\tau)^2\mathrm{d}\tau\leqslant\int_{t-1}^t\|\widetilde{v}(x,\tau),\widetilde{v}_t(x,\tau)\|^2_{\mathcal{H}}\mathrm{d}\tau+\psi(t)-\psi(t-1)\leqslant3\max_{\tau\in[t-1,t]}\|(\widetilde{v}(x,\tau),\widetilde{v}_t(x,\tau))\|^2_{\mathcal{H}}.\tag{26}$$

又因为

$$\int_0^1\widetilde{v}_t^2(0,t-x)\mathrm{d}x=\int_{t-1}^t\widetilde{v}_t^2(0,\tau)\mathrm{d}\tau,\tag{27}$$

因此, 式(22)得以证明. 接下来, 对系统(15)的W部分求解可得

$$W(x,t)=\begin{cases}\dfrac{-c_0}{1-c_0}\widetilde{v}(0,t-x),\ t\geqslant x,\\ W_0(x-t),\qquad x>t.\end{cases}\tag{28}$$

由Sobolev嵌入定理、式(15)(21), 可得

$$|W(0,t)|=\left|\frac{c_0}{1-c_0}\widetilde{v}(0,t)\right|\leqslant\left\|\frac{c_0}{1-c_0}\widetilde{v}(0,t)\right\|_{H^1(0,1)}\leqslant\frac{c_0}{1-c_0}\|\widetilde{v}(\cdot,t),\widetilde{v}_t(\cdot,t)\|_{\mathcal{H}}\leqslant\frac{c_0}{1-c_0}M_1.\tag{29}$$

对于$\forall t \geqslant 0$, 利用式(28)可得

$$\int_0^1 W_x^2(x,t)\mathrm{d}x = (\frac{c_0}{1-c_0})^2 \int_0^1 \widetilde{v}_t^2(0,t-x)\mathrm{d}x, \tag{30}$$

结合式(21)–(22)(29)可得$\|W(\cdot,t)\|_{H^1(0,1)}$的有界性. 因为$\widehat{v}(x,t) = \widetilde{v}(x,t) - W(x,t), W_t(x,t) = -W_x(x,t)$, 故

$$\begin{aligned}&\sup_{t\geqslant 0}\|(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t))\|_{\mathcal{H}} \leqslant \\ &\sup_{t\geqslant 0}(\|(\widetilde{v}(\cdot,t),\widetilde{v}_t(\cdot,t))\|_{\mathcal{H}} + \|(W(\cdot,t),W_t(\cdot,t))\|_{\mathcal{H}}).\end{aligned} \tag{31}$$

综上所述, 可以证明式(17). 进一步, 若$f(w,w_t)\in L^2(0,\infty)$, 或$\lim\limits_{t\to\infty}|f(w,w_t)|=0$, $d\in L^2(0,\infty)$, $h(t)\in L^2(0,\infty)$, 可由文献[16]中的引理2.1得到

$$\lim_{t\to\infty}\|(\widetilde{v}(\cdot,t),\widetilde{v}_t(\cdot,t))\|_{\mathcal{H}} = 0. \tag{32}$$

结合式(22)(32)得$\int_0^1 \widetilde{v}_t^2(0,t-x)\mathrm{d}x \to 0$(当$t\to\infty$). 因此, 由$\|W(\cdot,t)\|^2_{H^1(0,1)} = |W(0,t)|^2 + \int_0^1 W_x(x,t)\mathrm{d}x = (\frac{c_0}{1-c_0})^2(\widetilde{v}(0,t) + \int_0^1 \widetilde{v}_t^2(0,t-x)\mathrm{d}x)$, 可以推出$\lim\limits_{t\to\infty}\|W(\cdot,t)\|_{H^1(0,1)}=0$. 类似式(31)可得式(18). 证毕.

系统(13)的z部分被用来估计总扰动$F(t)$, 令$\widetilde{z}(x,t) = z(x,t) - \widetilde{v}(x,t)$, 则扰动误差系统$\widetilde{v}(x,t)$满足下式:

$$\begin{cases}\widetilde{z}_{xx}(x,t) = \widetilde{z}_{tt}(x,t),\\ \widetilde{z}_x(0,t) = \dfrac{c_0-q}{1-c_0}\widetilde{z}_t(0,t) + \dfrac{c_1}{1-c_0}\widetilde{z}(0,t),\\ \widetilde{z}(1,t) = 0.\end{cases} \tag{33}$$

在状态空间$\mathcal{H}_0 = H_R^1(0,1)\times L^2(0,1)$中研究系统(33), 其 中$H_R^1(0,1)=\{f\in H^1(0,1): f(1)=0\}$. 系统(33)可以写成如下算子形式:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{z}(.,t),\widetilde{z}_t(.,t)) = \mathcal{A}_{\widetilde{z}}(\widetilde{z}(.,t),\widetilde{z}_t(.,t)), \tag{34}$$

其中

$$\begin{cases}\mathcal{A}_{\widetilde{z}}(f,g) = (g,f''),\\ D(\mathcal{A}_{\widetilde{z}}) = \{(f,g)\in H^2(0,1)\times H^1(0,1)|\\ \qquad f'(0) = \dfrac{c_0-q}{1-c_0}g(0) + \dfrac{c_1}{1-c_0}f(0),\\ \qquad f(1)=g(1)=0\}.\end{cases} \tag{35}$$

显然$\mathcal{A}_{\widetilde{z}}$生成指数稳定的半群$\mathrm{e}^{\mathcal{A}_{\widetilde{z}}t}$, 因此, 对任意初值$(\widetilde{z}_0,\widetilde{z}_1)\in\mathcal{H}_0$, 系统(33)存在唯一解$(\widetilde{z}(.,t),\widetilde{z}_t(.,t)) = \mathrm{e}^{\mathcal{A}_{\widetilde{z}}t}(\widetilde{z}_0,\widetilde{z}_1)\in C(0,\infty;\mathcal{H}_0)$满足

$$\int_0^1(\widetilde{z}_x(x,t))^2 + (\widetilde{z}_t(x,t))^2\mathrm{d}x \leqslant M\|(\widetilde{z}_0,\widetilde{z}_1)\|^2_{\mathcal{H}_0}\mathrm{e}^{-\mu t}.$$

引理 2　设$\dfrac{c_0-q}{1-c_0}>0$, $\dfrac{c_1}{1-c_0}>0$, $\forall(\widetilde{z}_0,\widetilde{z}_1)\in\mathcal{H}_0$, 系统(33)的解满足

$$\int_0^\infty \widetilde{z}_x^2(1,t)\mathrm{d}x < \infty.$$

证　假设$\rho(x,t) = \int_0^1 x\widetilde{z}_t(x,t)\widetilde{z}_x(x,t)\mathrm{d}x$, 那么

$$\begin{aligned}|\rho(x,t)| &\leqslant \frac{1}{2}\int_0^1(\widetilde{z}_x(x,t))^2 + (\widetilde{z}_t(x,t))^2\mathrm{d}x \leqslant \\ &\frac{1}{2}M\|(\widetilde{z}_0,\widetilde{z}_1)\|^2_{\mathcal{H}_0}\mathrm{e}^{-\mu t}.\end{aligned} \tag{36}$$

根据式(33)对ρ求导得

$$\begin{aligned}&\frac{\partial}{\partial t}\rho(x,t) = \\ &\frac{1}{2}\widetilde{z}_x^2(1,t) - \frac{1}{2}\int_0^1(\widetilde{z}_x(x,t))^2 + (\widetilde{z}_t(x,t))^2\mathrm{d}x,\end{aligned} \tag{37}$$

上式关于t在$[0,T]$上积分, 可得

$$\begin{aligned}&\frac{1}{2}\int_0^T(\widetilde{z}_x(1,t))^2\mathrm{d}t = \\ &\frac{1}{2}\int_0^T\int_0^1(\widetilde{z}_x(x,t))^2 + (\widetilde{z}_t(x,t))^2\mathrm{d}x\mathrm{d}t + \rho(T) - \rho(0) \leqslant \\ &\frac{1}{2\mu}M\|(\widetilde{z}_0,\widetilde{z}_1)\|^2_{\mathcal{H}_0}(1-\mathrm{e}^{-\mu T}) + \\ &\frac{1}{2}M\|(\widetilde{z}_0,\widetilde{z}_1)\|^2_{\mathcal{H}_0}\mathrm{e}^{-\mu T} + \frac{1}{2}M\|(\widetilde{z}_0,\widetilde{z}_1)\|^2_{\mathcal{H}_0},\end{aligned} \tag{38}$$

当$T\to\infty$时, $\int_0^T(\widetilde{z}_x(1,t))^2\mathrm{d}t < \infty$, 所以$\widetilde{z}_x(1,t)\in L^2(0,\infty)$.　证毕.

推 论 1　假 设$(\widetilde{z}(\cdot,0),\widetilde{z}_t(\cdot,0))\in D(\mathcal{A}_{\widetilde{z}})$, 那 么, 系统(33)存在唯一经典解$(\widetilde{z},\widetilde{z}_t)\in C(0,\infty;D(\mathcal{A}_{\widetilde{z}}))$满足$|\widetilde{z}_x(1,t)| \leqslant Z_0L\mathrm{e}^{-w_{\mathcal{A}_{\widetilde{z}}}t}, \forall t\geqslant 0$, 其中$Z_0$是依赖于初值的正常数[17].

4　观测器设计与分析

在这一部分, 为系统(1)设计如下基于扰动估计量的状态观测器:

$$\begin{cases}\widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t),\\ \widehat{w}_x(0,t) = -q\widehat{w}_t(0,t) + p\widehat{w}^3(0,t) + \\ \qquad c_1(\widehat{w}(0,t) - w(0,t)),\\ \widehat{w}_x(1,t) = u(t) + \widehat{F}(t) - E_x(1,t),\\ \widehat{w}(x,0) = \widehat{w}_0(x), \widehat{w}_t(x,0) = \widehat{w}_1(x),\\ E_t(x,t) + E_x(x,t) = 0,\\ E(0,t) = -c_0(\widehat{w}(0,t) - w(0,t)),\\ E(x,0) = E_0(x),\end{cases} \tag{39}$$

信号$\widehat{F}(t) = z_x(1,t)$是由$z$子系统产生, 用来补偿总扰动$F(t)$, 为了证明观测器的渐近收敛性, 令$e(x,t) = \widehat{w}(x,t) - w(x,t)$, 记$h_0(\cdot,t) = p(\widehat{w}^3(0,t) - w^3(0,t))$, 可得

$$\begin{cases} e_{tt}(x,t) = e_{xx}(x,t), \\ e_x(0,t) = -qe_t(0,t) + c_1 e(0,t) + h_0(\cdot,t), \\ e_x(1,t) = \widetilde{z}_x(1,t) - E_x(1,t), \\ E_t(x,t) + E_x(x,t) = 0, \\ E(0,t) = -c_0 e(0,t). \end{cases} \tag{40}$$

引理 3 设$\frac{c_0-q}{1-c_0}>0$, $\frac{c_1}{1-c_0}>0$, 信号$\widetilde{z}_x(1,t)$由系统(33)产生, 对于任何初值$(e(\cdot,0), e_t(\cdot,0), E(\cdot,0)) \in \mathcal{H}\times H^1(0,1)$, 且满足相容性条件$E_0(0) = -c_0 e_0(0)$, 那么, 系统(40)存在唯一解$(e, e_t, E) \in C(0,\infty;\mathcal{H}\times H^1(0,1))$满足

$$\lim_{t\to\infty} \|(e,e_t,E)\|_{\mathcal{H}\times H^1(0,1)} = 0. \tag{41}$$

证 令$\widetilde{e}(x,t) = e(x,t)+E(x,t)$, 那么, $(\widetilde{e}, E)$系统变为如下形式:

$$\begin{cases} \widetilde{e}_{tt}(x,t) = \widetilde{e}_{xx}(x,t), \\ \widetilde{e}_x(0,t) = \frac{c_0-q}{1-c_0}\widetilde{e}_t(0,t) + \frac{c_1}{1-c_0}\widetilde{e}(0,t) + h_0(\cdot,t), \\ \widetilde{e}_x(1,t) = \widetilde{z}_x(1,t), \\ \widetilde{e}(x,0) = \widetilde{e}_0(x),\ \widetilde{e}_t(x,0) = \widetilde{e}_1(x), \\ E_t(x,t) + E_x(x,t) = 0, \\ E(0,t) = -\frac{c_0}{1-c_0}\widetilde{e}(0,t), \end{cases} \tag{42}$$

其中

$$\widetilde{e}(x,0) = e_0(x) + E_0(x),\ \widetilde{e}_t(x,0) = e_1(x) - E_{0x}(x). \tag{43}$$

系统(42)可以写成如下抽象发展方程的形式:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{e}(\cdot,t), \widetilde{e}_t(\cdot,t)) = \mathcal{A}_{\widetilde{e}}(\widetilde{e}(\cdot,t), \widetilde{e}_t(\cdot,t)) + \mathcal{B}_1 h_0(\cdot,t) + \mathcal{B}_2 \widetilde{z}_x(1,t), \tag{44}$$

其中

$$\begin{cases} \mathcal{A}_{\widetilde{e}}(f,g) = (g, f''), \\ D(\mathcal{A}_{\widetilde{e}}) = \{(f,g)\in H^2(0,1)\times H^1(0,1) | \\ \qquad f'(0) = \frac{c_0-q}{1-c_0}g(0) + \frac{c_1}{1-c_0}f(0), \\ \qquad f'(1) = 0\}, \\ \mathcal{B}_1 = (0, -\delta(x)),\ \mathcal{B}_2 = (0, \delta(x-1)). \end{cases} \tag{45}$$

类似引理1的证明, 容易推得$\mathcal{A}_{\widetilde{e}}$生成了指数稳定的$C_0$半群$\mathrm{e}^{\mathcal{A}_{\widetilde{e}}t}$, $\mathcal{B}_1$和$\mathcal{B}_2$是可允许算子. 由$h_0(\cdot,t) \in L^2(0,\infty)$、引理2以及文献[16]的引理2.1, 系统(42)存在唯一的渐进稳定解满足$\lim_{t\to\infty}\|(\widetilde{e},\widetilde{e}_t)\|_{\mathcal{H}} = 0$. 接下来, 利用特征线方法可得

$$E(x,t) = \begin{cases} \frac{-c_0}{1-c_0}\widetilde{e}(0,t-x),\ t \geqslant x, \\ E_0(x-t),\qquad x > t. \end{cases} \tag{46}$$

根据引理1中$W(x,t)$的渐近稳定性, 故有$\lim_{t\to\infty}\|E(\cdot,t)\|_{H^1(0,1)} = 0$. 由于$e(x,t) = \widetilde{e}(x,t) - E(x,t)$和$E_t(x,t) = -E_x(x,t)$, 因此, $\|(e(\cdot,t), e(\cdot,t))\|_{\mathcal{H}} \leqslant \|(\widetilde{e}(\cdot,t), \widetilde{e}_t(\cdot,t))\|_{\mathcal{H}} + \|(E(\cdot,t), E_t(\cdot,t))\|_{\mathcal{H}} \to 0$, 当$t\to\infty$. 证毕.

5 基于观测器的输出反馈控制

在这一部分, 为系统(1)设计控制律, 因此, 需要首先考虑系统(39)的稳定控制律, 为此引入一个辅助系统

$$\begin{cases} \widehat{W}_t(x,t) + \widehat{W}_x(x,t) = 0, \\ \widehat{W}(0,t) = -c_2\widehat{w}(0,t),\ \widehat{W}(x,0) = \widehat{W}_0(x). \end{cases} \tag{47}$$

接下来, 令$\widetilde{w}(x,t) = \widehat{w}(x,t) + \widehat{W}(x,t)$, 那么$(\widetilde{w}, \widehat{W})$满足

$$\begin{cases} \widetilde{w}_{tt}(x,t) = \widetilde{w}_{xx}(x,t), \\ \widetilde{w}_x(0,t) = \\ \frac{c_2-q}{1-c_2}\widetilde{w}_t(0,t) + \frac{p}{(1-c_2)^3}\widetilde{w}^3(0,t) + c_1 e(0,t), \\ \widetilde{w}_x(1,t) = u(t) + z_x(1,t) - E_x(1,t) + \widehat{W}_x(1,t), \\ \widehat{W}_t(x,t) + \widehat{W}_x(x,t) = 0, \\ \widehat{W}(0,t) = -\frac{c_2}{1-c_2}\widetilde{w}(0,t). \end{cases} \tag{48}$$

令$\frac{c_2-q}{1-c_2} > 0$, 由此可以看出, 上述方程左端具有阻尼项. 设计反馈控制如下:

$$u(t) = -c_3\widetilde{w}(1,t) - z_x(1,t) + E_x(1,t) - \widehat{W}_x(1,t), \tag{49}$$

由此得到如下闭环系统:

$$\begin{cases} \widetilde{w}_{tt}(x,t) = \widetilde{w}_{xx}(x,t), \\ \widetilde{w}_x(0,t) = \\ \frac{c_2-q}{1-c_2}\widetilde{w}_t(0,t) + \frac{p}{(1-c_2)^3}\widetilde{w}^3(0,t) + c_1 e(0,t), \\ \widetilde{w}_x(1,t) = -c_3\widetilde{w}(1,t), \\ \widehat{W}_t(x,t) + \widehat{W}_x(x,t) = 0, \\ \widehat{W}(0,t) = -\frac{c_2}{1-c_2}\widetilde{w}(0,t). \end{cases} \tag{50}$$

结合式(33)(40)(50), 有

$$\begin{cases} e_{tt}(x,t) = e_{xx}(x,t), \\ e_x(0,t) = -qe_t(0,t) + c_1 e(0,t) + h_0(\cdot,t), \\ e_x(1,t) = \widetilde{z}_x(1,t) - E_x(1,t), \\ E_t(x,t) + E_x(x,t) = 0,\ E(0,t) = -c_0 e(0,t), \end{cases}$$

$$\begin{cases}\widetilde{z}_{xx}(x,t)=\widetilde{z}_{tt}(x,t),\\ \widetilde{z}_x(0,t)=\dfrac{c_0-q}{1-c_0}\widetilde{z}_t(0,t)+\dfrac{c_1}{1-c_0}\widetilde{z}(0,t),\\ \widetilde{z}(1,t)=0,\\ \widetilde{w}_{tt}(x,t)=\widetilde{w}_{xx}(x,t),\\ \widetilde{w}_x(0,t)=\\ \dfrac{c_2-q}{1-c_2}\widetilde{w}_t(0,t)+\dfrac{p}{(1-c_2)^3}\widetilde{w}^3(0,t)+c_1e(0,t),\\ \widetilde{w}_x(1,t)=-c_3\widetilde{w}(1,t),\\ \widehat{W}_t(x,t)+\widehat{W}_x(x,t)=0,\\ \widehat{W}(0,t)=-\dfrac{c_2}{1-c_2}\widetilde{w}(0,t),\end{cases}\tag{51}$$

接下来在状态空间$\mathcal{X}=\mathcal{H}\times H^1(0,1)\times H_R^1(0,1)\times L^2(0,1)\times\mathcal{H}\times H^1(0,1)$中考虑系统(51).

定理 1　假设 $\dfrac{c_2-q}{1-c_2}>0$, $\dfrac{p}{(1-c_2)^3}>0$, $\dfrac{c_0-q}{1-c_0}>0$, $\dfrac{c_1}{1-c_0}>0$, $c_3>0$, 以及引理 3 中的假设也成立, 那么, 对于任意初值$(e_0,e_1,E_0,\widetilde{z}_0,\widetilde{z}_1,\widetilde{w}_0,\widetilde{w}_1,\widehat{W})\in\mathcal{X}$, 以及满足相容性条件$E_0(0)=-c_0e_0(0)$, $\widehat{W}_0(0)=-\dfrac{c_2}{1-c_2}\widetilde{w}_0(0)$, 那么, 系统(51)存在唯一解$(e,e_t,E,\widetilde{z},\widetilde{z}_t,\widetilde{w},\widetilde{w}_t,\widehat{W})\in C(0,\infty;\mathcal{X})$, 并且满足

$$\lim_{t\to\infty}\|(e(\cdot,t),e_t(\cdot,t),E(\cdot,t),\widetilde{z}(\cdot,t),\widetilde{z}_t(\cdot,t),\widetilde{w}(\cdot,t),\widetilde{w}_t(\cdot,t),\widehat{W}(\cdot,t))\|_{\mathcal{X}}=0.\tag{52}$$

证　首先考虑系统(51)中的$(\widetilde{e},\widetilde{z})$部分, 由引理3中引入的已知变换$\widetilde{e}(x,t)=e(x,t)+E(x,t)$, 容易验证$(\widetilde{e},\widetilde{z})$ 满足

$$\begin{cases}\widetilde{e}_{tt}(x,t)=\widetilde{e}_{xx}(x,t),\\ \widetilde{e}_x(0,t)=\dfrac{c_0-q}{1-c_0}\widetilde{e}_t(0,t)+\dfrac{c_1}{1-c_0}\widetilde{e}(0,t)+h_0(.,t),\\ \widetilde{e}_x(1,t)=\widetilde{z}_x(1,t),\\ E_t(x,t)+E_x(x,t)=0,\\ E(0,t)=-\dfrac{c_0}{1-c_0}\widetilde{e}(0,t),\\ \widetilde{z}_{xx}(x,t)=\widetilde{z}_{tt}(x,t),\\ \widetilde{z}_x(0,t)=\dfrac{c_0-q}{1-c_0}\widetilde{z}_t(0,t)+\dfrac{c_1}{1-c_0}\widetilde{z}(0,t),\\ \widetilde{z}(1,t)=0,\end{cases}\tag{53}$$

系统(53)可以写成如下算子形式:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{e}(\cdot,t),\widetilde{e}_t(\cdot,t),\widetilde{z}(\cdot,t),\widetilde{z}_t(\cdot,t))=\mathcal{A}_{\widetilde{e}\widetilde{z}}(\widetilde{e}(\cdot,t),\widetilde{e}_t(\cdot,t),\widetilde{z}(\cdot,t),\widetilde{z}_t(\cdot,t))+\mathcal{B}_3h_0(\cdot,t)+\mathcal{B}_4\widetilde{z}_x(1,t),\tag{54}$$

其中

$$\begin{cases}\mathcal{A}_{\widetilde{e}\widetilde{z}}(f,g,\phi,\psi)=(g,f'',\psi,\phi''),\\ D(\mathcal{A}_{\widetilde{e}\widetilde{z}})=\{(f,g,\phi,\psi)\in(H^2(0,1)\times H^1(0,1))^2|\\ \quad f'(0)=\dfrac{c_0-q}{1-c_0}g(0)+\dfrac{c_1}{1-c_0}f(0),\\ \quad f'(1)=0,\ \phi'(0)=\dfrac{c_0-q}{1-c_0}\psi(0)+\\ \quad \dfrac{c_1}{1-c_0}\phi(0),\ \phi(1)=0\},\\ \mathcal{B}_3=(0,-\delta(x),0,0),\ \mathcal{B}_4=(0,\delta(x-1),0,0),\end{cases}\tag{55}$$

又因为算子$\mathcal{A}_{\widetilde{e}\widetilde{z}}$生成了$C_0$半群, $\mathcal{B}_3$, $\mathcal{B}_4$是可允许的, 那么, 由引理3可得

$$\lim_{t\to\infty}\|(\widetilde{e}(\cdot,t),\widetilde{e}_t(\cdot,t),\widetilde{z}(\cdot,t),\widetilde{z}_t(\cdot,t))\|=0,\tag{56}$$

类似于引理3的证明可以推得$\lim\limits_{t\to\infty}\|(e(\cdot,t),e_t(\cdot,t),E(\cdot,t))\|_{\mathcal{H}\times H^1(0,1)}=0$, 从而

$$\lim_{t\to\infty}\|(e(\cdot,t),e_t(\cdot,t),\widetilde{z}(\cdot,t),\widetilde{z}_t(\cdot,t),E(\cdot,t))\|=0.\tag{57}$$

接下来研究系统(51)的$(\widetilde{w},\widehat{W})$部分

$$\begin{cases}\widetilde{w}_{tt}(x,t)=\widetilde{w}_{xx}(x,t),\\ \widetilde{w}_x(0,t)=\\ \dfrac{c_2-q}{1-c_2}\widetilde{w}_t(0,t)+\dfrac{p}{(1-c_2)^3}\widetilde{w}^3(0,t)+c_1e(0,t),\\ \widetilde{w}_x(1,t)=-c_3\widetilde{w}(1,t),\\ \widehat{W}_t(x,t)+\widehat{W}_x(x,t)=0,\\ \widehat{W}(0,t)=-\dfrac{c_2}{1-c_2}\widetilde{w}(0,t).\end{cases}\tag{58}$$

在空间$\mathcal{H}=H^1(0,1)\times L^2(0,1)$中考虑系统(58). 定义内积

$$\langle(f_1,g_1),(f_2,g_2)\rangle=\int_0^1(f_1'(x)\bar{f}_2'(x)+g_1(x)\bar{g}_2(x))\mathrm{d}x+c_3f_1(1)\bar{f}_2(1),\ \forall(f_i,g_i)\in\mathcal{H},\ i=1,2.\tag{59}$$

由 Sobolev 嵌入定理和上述结论 (58), 容易推得$|e(0,t)|\leqslant\|e(0,t)\|_{H^1(0,1)}\leqslant\|(e(\cdot,t),\ e_t(\cdot,t))\|_{\mathcal{H}}\to 0(t\to\infty)$. 将系统(58)改写成如下形式:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{w}(\cdot,t),\widetilde{w}_t(\cdot,t))=\mathcal{A}_{\widetilde{w}}(\widetilde{w}(\cdot,t),\widetilde{w}_t(\cdot,t))+\mathcal{B}\Big(\frac{p}{(1-c_2)^3}\widetilde{w}^3(0,t)+c_1e(0,t)\Big),\tag{60}$$

其中

$$\begin{cases}\mathcal{A}_{\widetilde{w}}(f,g)=(g,f''),\\ D(\mathcal{A}_{\widetilde{w}})=\{(f,g)\in H^2(0,1)\times H^1(0,1)|\\ \qquad f'(0)=\dfrac{c_2-q}{1-c_2}g(0),\\ \qquad f'(1)=-c_3f(1)\},\\ \mathcal{B}=(0,\delta(x)).\end{cases}\tag{61}$$

根据文献[12]中的定理4.1可知, $\mathcal{A}_{\widetilde{w}}$生成了$C_0$压缩半群$\mathrm{e}^{\mathcal{A}_{\widetilde{w}}t}$, $\mathcal{B}$是可允许算子, 因此, 根据文献[16]的引理2.1得

$$\lim_{t\to\infty}\|(\widetilde{w}(\cdot,t),\widetilde{w}_t(\cdot,t))\|=0.\tag{62}$$

接下来, 证明系统(58)中$\widehat{W}$的解是渐近稳定的. 令

$$\widehat{W}(x,t)=\begin{cases}\dfrac{-c_2}{1-c_2}\widetilde{w}(0,t-x),\ t\geqslant x,\\ \widehat{W}_0(x-t),\qquad\qquad x>t.\end{cases}\tag{63}$$

为了证明$\widehat{W}$部分解是渐近稳定性的, 需要证明

$$\lim_{t\to\infty}\int_0^1\widetilde{w}_t^2(0,t-x)\mathrm{d}x=0\tag{64}$$

成立. 定义$\rho(t)=2\int_0^1(x-1)\widetilde{w}_t(x,t)\mathrm{d}x$, 式(64)的其余证明过程类似于式(22)的证明, 故在这里略去详细过程. 结合式(57)(62)–(63)和$\widehat{W}(x,t)$在$H^1(0,1)$的渐近稳定性, 可以得到结论式(52)成立. 证毕.

接下来证明闭环系统的稳定性, 基于反馈控制(49), 可得如下闭环系统:

$$\begin{cases}w_{tt}(x,t)=w_{xx}(x,t),\\ w_x(0,t)=-qw_t(0,t)+pw(0,t)^3,\\ w_x(1,t)=-c_3\widehat{w}(1,t)-c_3\widehat{W}(1,t)-z_x(1,t)+\\ \qquad E_x(1,t)-\widehat{W}_x(1,t)+d(t)+\\ \qquad f(w(\cdot,t),w_t(\cdot,t)),\\ v_{xx}(x,t)=v_{tt}(x,t),\\ v_x(0,t)=\\ -qv_t(0,t)+pv^3(0,t)+c_1(v(0,t)-w(0,t)),\\ v_x(1,t)=-c_3\widehat{w}(1,t)-c_3\widehat{W}(1,t)-z_x(1,t)+\\ \qquad E_x(1,t)-\widehat{W}_x(1,t)+W_x(1,t),\\ z_{xx}(x,t)=z_{tt}(x,t),\\ z_x(0,t)=\dfrac{c_0-q}{1-c_0}z_t(0,t)+\dfrac{c_1}{1-c_0}z(0,t)+h(\cdot,t),\\ z(1,t)=w(1,t)-v(1,t)+W(1,t),\\ W_t(x,t)+W_x(x,t)=0,\\ W(0,t)=-c_0(w(0,t)-v(0,t)),\\ \widehat{w}_{tt}(x,t)=\widehat{w}_{xx}(x,t),\\ \widehat{w}_x(0,t)=\\ -q\widehat{w}_t(0,t)+p\widehat{w}^3(0,t)+c_1(\widehat{w}(0,t)-w(0,t)),\\ \widehat{w}_x(1,t)=-c_3\widehat{w}(1,t)-c_3\widehat{W}(1,t)-\widehat{W}_x(1,t),\\ E_t(x,t)+E_x(x,t)=0,\\ E(0,t)=-c_0(\widehat{w}(0,t)-w(0,t)),\\ \widehat{W}_t(x,t)+\widehat{W}_x(x,t)=0,\\ \widehat{W}(0,t)=-c_2\widehat{w}(0,t),\end{cases}\tag{65}$$

在空间$\mathcal{H}_1=\mathcal{H}^3\times H^1(0,1)\times\mathcal{H}\times(H^1(0,1))^2$中考虑系统(65).

定理 2 假设$\dfrac{c_0-q}{1-c_0}>0$, $\dfrac{c_2-q}{1-c_2}>0$, $\dfrac{p}{(1-c_2)^3}>0$, 以及$f:\mathcal{H}\to\mathbb{R}$是连续的, 且$d\in L^\infty(0,\infty)$或$d\in L^2(0,\infty)$. 对于任意初始状态$(w_0,w_1,v_0,v_1,z_0,z_1,W_0,\widehat{w}_0,\widehat{w}_1,E_0,\widehat{W}_0)\in\mathcal{H}_1$, 且满足相容性条件$z_0(1)=w_0(1)-v_0(1)+W_0(1)$, $\widehat{W}_0(0)=-c_2\widehat{w}_0(0)$, $W_0(0)=-c_0(w_0(0)-v_0(0))$, $E_0=-c_0(\widehat{w}_0(0)-w_0(0))$, 那么, 系统(65)存在唯一解$(w,w_t,v,v_t,z,z_t,W,\widehat{w},\widehat{w}_t,E,\widehat{W})$且满足

$$\begin{aligned}&\lim_{t\to\infty}\|(w(\cdot,t),w_t(\cdot,t),\widehat{w}(\cdot,t),\\&\widehat{w}_t(\cdot,t),E(\cdot,t),\widehat{W}(\cdot,t))\|_{\mathcal{H}^2\times(H^1(0,1))^2}=0\end{aligned}\tag{66}$$

和

$$\sup_{t\geqslant0}\|(v(\cdot,t),v_t(\cdot,t),z(\cdot,t),z_t(\cdot,t),W(\cdot,t))\|<\infty.\tag{67}$$

如果进一步假设$f(0,0)=0$和$d\in L^2(0,\infty)$, 那么

$$\begin{aligned}&\lim_{t\to\infty}\|(v(\cdot,t),v_t(\cdot,t),z(\cdot,t),z_t(\cdot,t),\\&W(\cdot,t))\|_{\mathcal{H}^2\times(H^1(0,1))^2}=0.\end{aligned}\tag{68}$$

证 令$e(x,t)=\widehat{w}(x,t)-w(x,t)$, $\widetilde{z}(x,t)=z(x,t)-\widetilde{v}(x,t)$, $\widehat{v}(x,t)=w(x,t)-v(x,t)$, $\widetilde{w}(x,t)=\widehat{w}(x,t)+\widehat{W}(x,t)$.

可得如下变换后的系统:

$$\begin{cases}e_{tt}(x,t)=e_{xx}(x,t),\\ e_x(0,t)=-qe_t(0,t)+c_1e(0,t)+h_0(.,t),\\ e_x(1,t)=\widetilde{z}_x(1,t)-E_x(1,t),\\ E_t(x,t)+E_x(x,t)=0,\ E(0,t)=-c_0e(0,t),\\ \widetilde{z}_{xx}(x,t)=\widetilde{z}_{tt}(x,t),\\ \widetilde{z}_x(0,t)=\dfrac{c_0-q}{1-c_0}\widetilde{z}_t(0,t)+\dfrac{c_1}{1-c_0}\widetilde{z}(0,t),\\ \widetilde{z}(1,t)=0,\end{cases}$$

$$\begin{cases}\widetilde{w}_{tt}(x,t)=\widetilde{w}_{xx}(x,t),\\ \widetilde{w}_x(0,t)=\\ \dfrac{c_2-q}{1-c_2}\widetilde{w}_t(0,t)+\dfrac{p}{(1-c_2)^3}\widetilde{w}^3(0,t)+c_1e(0,t),\\ \widetilde{w}_x(1,t)=-c_3\widetilde{w}(1,t),\\ \widehat{W}_t(x,t)+\widehat{W}_x(x,t)=0,\\ \widehat{W}(0,t)=-\dfrac{c_2}{1-c_2}\widetilde{w}(0,t),\\ \widehat{v}_{xx}(x,t)=\widehat{v}_{tt}(x,t),\\ \widehat{v}_x(0,t)=-q\widehat{v}_t(0,t)+c_1\widehat{v}(0,t)+h(.,t),\\ \widehat{v}_x(1,t)=F(t)-W_x(1,t),\\ W_t(x,t)+W_x(x,t)=0,\ W(0,t)=-c_0\widehat{v}(0,t).\end{cases} \tag{69}$$

该系统等价于闭环系统(65). 显然, 系统(69)中的$(e,E,\widetilde{z},\widetilde{w},\widehat{W})$子系统独立于$(\widehat{v},W)$, 那么由定理1可得, $(e,e_t,E,\widetilde{z},\widetilde{z}_t,\widetilde{w},\widetilde{w}_t,\widehat{W})\in C(0,\infty;\mathcal{X})$, 且满足

$$\lim_{t\to\infty}\|(e(\cdot,t),e_t(\cdot,t),E(\cdot,t),\widetilde{z}(\cdot,t),\widetilde{z}_t(\cdot,t),\\ \widetilde{w}(\cdot,t),\widetilde{w}_t(\cdot,t),\widehat{W}(\cdot,t))\|_{\mathcal{X}}=0. \tag{70}$$

$$\begin{bmatrix}w\\ w_t\\ \widehat{w}\\ \widehat{w}_t\end{bmatrix}=\begin{bmatrix}-I & 0 & (I+P)^{-1} & 0\\ 0 & -I & 0 & (I+P)^{-1}\\ 0 & 0 & (I+P)^{-1} & 0\\ 0 & 0 & 0 & (I+P)^{-1}\end{bmatrix}\begin{bmatrix}e\\ e_t\\ \widetilde{w}\\ \widetilde{w}_t\end{bmatrix}, \tag{71}$$

其中$\widetilde{w}(x,t)=(I+P)\widehat{w}(x,t)=\widehat{w}(x,t)+\widehat{W}(x,t)$, 由$e(x,t)=\widehat{w}(x,t)-w(x,t)$, 以及式(69)–(71)得

$$\lim_{t\to\infty}\|(w(\cdot,t),w_t(x,t),\widehat{w}(x,t),\widehat{w}_t(x,t))\|=0. \tag{72}$$

接下来考虑系统(69)的$(\widehat{v},W)$部分. 因为$f:\mathcal{H}\to\mathbb{R}$连续, (w,w_t)有界, 所以可以得到$f(w(\cdot,t),w_t(\cdot,t))\in L^\infty(0,\infty)$. 又因为$d\in L^\infty(0,\infty)$或$L^2(0,\infty)$, $h(\cdot,t)\in L^2(0,\infty)$, 由引理2, 可知$(\widehat{v},W))$存在唯一有界解,

$$\sup_{t\geqslant 0}\|(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t),W(\cdot,t))\|_{\mathcal{H}\times H^1(0,1)}<\infty. \tag{73}$$

根据变换$\widehat{v}(x,t)=v(x,t)-w(x,t)$, $\widetilde{z}(x,t)=z(x,t)-\widetilde{v}(x,t)$以及范数性质, 可得

$$\|(v(\cdot,t),v_t(\cdot,t))\|_{\mathcal{H}}\leqslant\\ \|(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t))\|_{\mathcal{H}}+\|(w(\cdot,t),w_t(\cdot,t))\|_{\mathcal{H}},\\ \|(z(\cdot,t),z_t(\cdot,t))\|_{\mathcal{H}}\leqslant\\ \|(\widetilde{z}(\cdot,t),\widetilde{z}_t(\cdot,t))\|_{\mathcal{H}}+\|(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t))\|_{\mathcal{H}}+\\ \|(W(\cdot,t),W_t(\cdot,t))\|_{\mathcal{H}},$$

故式(67)成立. 进一步假设$f(0,0)=0$和$d\in L^2(0,\infty)$, 由f的连续性, 有$\lim\limits_{t\to\infty}|f(w(\cdot,t),w_t(\cdot,t)|=0$. 再由引理2, 有$\lim\limits_{t\to\infty}\|(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t),W(\cdot,t))\|_{\mathcal{H}\times H^1(0,1)}=0$. 类似于上述的范数性质, 式(68)成立.　 证毕.

6 结论

本文研究了一类含有非线性边界条件的内部不确定性和外部扰动的一维波动方程的稳定性. 由于非线性位移边界条件和总扰动的存在, 系统具有复杂的动力学行为, 系统便变得不稳定. 首先, 本文设计了干扰估计器; 然后, 在扰动估计器的基础上提出了状态观测器. 结果表明, 由扰动估计器误差估计的总扰动属于$L^2(0,\infty)$, 且具有内部不确定性和外部扰动的非线性波动方程, 可以成功构造合适的控制器使得闭环系统渐近稳定.

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